W2L4 - HIGHER ORDER LINEAR ODES

Consider the n^{th} order IVP:

$$\begin{cases} a_n(x)\dfrac{d^n y}{dx_n} + a_{n-1}(x)\dfrac{d^{n-1}y}{dx_{n-1}} + \dots + a_1(x)\dfrac{dy}{dx} + a_0(x)\,y = g(x) \\ y(x_0)=y_0,\; y'(x_0)=y_1,\; \dots,\; y^{(n-1)}(x)=y_{n-1} \end{cases} \qquad (Eq.\,1)$$

Def: The linear n^{th} order ODE of the form in Eq. 1 is called homogeneous if $g(x)=0$ and is called non-homogenous if $g(x)\neq 0$.

Note: In order to solve a non-homogenous equation, we must first be able to solve the associated homogeneous equation:

$$a_n(x)\frac{d^n y}{dx_n} + a_{n-1}(x)\frac{d^{n-1}y}{dx_{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = 0 \qquad (Eq.\,2)$$

Right side is ZERO

Important Notation: Differential Operator

Recall: $Dy \equiv \dfrac{dy}{dx}$ or in other words $D = \dfrac{d}{dx}$

$D^2 = \dfrac{d^2}{dx^2}$, $D^3 = \dfrac{d^3}{dx^3}$, $\dots$, $D_n = \dfrac{d^n}{dx^n}$

EX: Consider $\underbrace{y''+3y'-4y}=0$

$D^2y + 3Dy - 4y = 0$

$\Rightarrow (D^2+3D-4)y = 0$

Def: the n^{th} order differential operator, or polynomail operator is given by

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_1(x)D + a_0(x)$$

$$Ly \Leftrightarrow \left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_1(x)D + a_0(x)\right)y$$

Eq 2 is equivalent to $Ly = g$

Def: an operator, L, is said to be linear if and only if

$$L(\alpha f(x) + \beta g(x)) = \alpha L(f(x)) + \beta L(g(x))$$

$L(\alpha f) = \alpha L(f)$

$L(f+g) = L(f) + L(g)$

EX: Consider the operator $L(y) = 2y$. Show that it is linear.

Proofs: $L(\alpha f + \beta g) = 2(\alpha f + \beta g)$

$= 2\alpha f + 2\beta g$

$= \alpha(2f) + \beta(2g) = \alpha L(f) + \beta L(g)$

Theorem: The differential operator is linear.

Proof: Let $L(y) = Dy$

wish to show: $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$

$D(\alpha f + \beta g) = D(\alpha f) + D(\beta g)$ (sum rule for derivative)

$= \alpha D(f) + \beta D(g)$ (scalar multi rule for derivative)

THE SUPERPOSITION PRINCIPLE

Consider the ODE:

$$x^3 y''' - 2xy' + 4y = 0$$

Show that $y_1 = x^2$ and $y_2 = x^2 \ln x$ in both solutions:

$y_1' = 2x$
$y_1'' = 2$
$y_1''' = 0$

$y_2' = x + 2x\ln x$
$y_2'' = 1 + 2 + 2\ln x = 3 + 2\ln x$
$y_2''' = 2/x$

$x^3(0) - 2x(2x) + 4(x^2) = 0$ ✓

$x^3\left(\frac{2}{x}\right) - 2x(x + 2x\ln x) + 4x^2\ln x = 2x^2 - 2x^2 - 4x^2\ln x + 4x^2\ln x = 0$ ✓

$$\Rightarrow \left.\begin{aligned} &x^3 y_1''' - 2xy_1' + 4y_1 = 0 \\ \wedge\ &x^3 y_2''' - 2xy_2' + 4y_2 = 0 \end{aligned}\right\}$$

and

Claim: $c_1 y_1 + c_2 y_2$ is also a solution.

Proof: $y' = c_1 y_1' + c_2 y_2'$, $y'' = c_1 y_1'' + c_2 y_2''$, $y''' = c_1 y_1''' + c_2 y_2'''$

Plug into the DE:

$$\Rightarrow x^3(c_1 y_1''' + c_2 y_2''') - 2x(c_1 y_1' + c_2 y_2') + 4(c_1 y_1 + c_2 y_2)$$

$$c_1\underbrace{(x^3 y_1''' - 2xy_1' + 4y_1)}_{=0} + c_2\underbrace{(x^3 y_2''' - 2xy_2' + 4y_2)}_{=0}$$

$c_1(0) + c_2(0) = 0$ ✓ $\rightarrow$ $y = c_1 y_1 + c_2 y_2$ is also a solution!

Theorem: (The Superposition Principal – Homogeneous Equations) – Let $y_1, y_2, \ldots, y_k$ be solutions of Eq. (2) over some interval I such that $x_0 \in I$. Then, any linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where $c_i \in \mathbb{R}\ \forall\ i = 1, 2, \ldots, k$ is also a solution to the Eq 2.

Corollary: 1. If y_1 is a solution then so is cy_1

2. $y = 0$ is always a solution to ANY homogeneous equation. → trivial solution

LINEAR DEPENDANCE AND INDEPENDANCE

Def: A set functions $f_1(x), f_2(x), \dots, f_n(x)$ are linearly dependent if there exists constants $c_1, c_2, \dots c_n$ that are not all zeros such that:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \qquad \text{Eq. 3}$$

If Eq. 3 has only the trivial solution for the c_i's (eg. they are all zeros) then the functions are called linearly independent.

EX: Let $f_1(x) = \cos^2(x)$, $f_2(x) = \sin^2(x)$, $f_3(x) = \sec^2(x)$, and $f_4(x) = \tan^2(x)$. Are these functions linearly dependent or independent?

$\Rightarrow$ Essentually this problem is a lot of work.
(see video for example: 45:35)

There has to be a better way!

THE WRONSKIAN (or "How to show functions are linearly independent")

Def: Suppose each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ posses $n-1$ derivatives. The Wronskian is given by:

$$W(f_1, f_2, \dots, f_n) = \det\left(\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \ddots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{bmatrix}\right)$$

Theorem: (Criterion for Linearly Independent Functions)
Let $y_1, y_2, \dots, y_n$ be n solutions of the Eq. 2 on some interval I. Then the set of solutions is linearly independent iff $W(y_1, y_2, \dots, y_n) \neq 0 \ \forall x \in I$.

EX: Let $y_1 = x^2$ and $y_2 = x^2 \ln x$. Are they linearly dependent or independent?

$$W = \det\left(\begin{bmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x\ln x \end{bmatrix}\right) = x^2(x + 2x\ln x) - (x^2 \ln x)(2x)$$
$$= x^3 + \cancel{2x^3 \ln x} - \cancel{2x^3 \ln x}$$
$$= x^3 \leftarrow \text{this is } 0 \text{ for } x = 0$$

$\Rightarrow \{x^2, x^2 \ln x\}$ are independent on any interval I not containing zero.

Def: Any set $y_1, y_2, \dots, y_n$ of n linearly independent solutions to the Eq. 2 on some interval I is said to be a fundamental set of solutions.
↑ homogeneous

Theorem: There exists a fundamental set of solutions for the Eq. 2 on some interval I on the condition that $y_i(x) \in C^0(I) \ \forall \ i = 1, \dots, n$.
↑ continuous

Theorem: Let $y_1, y_2, \dots y_n$ be a fundamental set of solutions. Then the general form of the solution to Eq. 2 is given by

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

← linear combination of your fundamental set.

Where c_i, for $i = 1, 2, \dots, n$ are arbitrary constants.

Proof: Eq 2 $\Leftrightarrow$ $Ly = 0$

Wish To Show: $\underbrace{L(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) = 0}$

$\hookrightarrow c_1 L(\cancel{y_1})^{0} + c_2 L(\cancel{y_2})^{0} + \cdots + c_n L(\cancel{y_n})^{0} = 0 \quad \square$

since y_i is also a solution to Eq. 2

EX: Given $y''' - 6y'' + 11y' - 6y = 0$, assume that $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are all solutions. (a) show that they form a fundamental set of solutions.

Wish To Show: Linearly Independent.

$$W(e^x, e^{2x}, e^{3x}) = \det\left(\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix}\right) = 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 9e^{6x} - 12e^{6x}$$

$= 2e^{6x} \neq 0 \quad \forall x$

$\Rightarrow$ linear independent on $(-\infty, \infty)$

$\Rightarrow$ Fundamental Set $\square$

(b) What is the general solution?

$$\underline{y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}$$

NON-HOMOGENEOUS EQUATIONS

Consider the non-homogeneous n^{th} order linear ODE:

$$a_n(x) \frac{d^n y}{dx_n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \qquad (Eq. 4)$$

where $g(x) \neq 0$.

Theorem: Let y_p be any particular solution to Eq 4. Let $y_1, y_2, \dots, y_n$ be solutions to the associated homogeneous equation, e.g., Eq. 2. Then, the general solution to Eq. 4 is given by:

$$y = \underbrace{c_1 y_1 + c_2 y_2 + \cdots + c_n y_n}_{y_h} + \underbrace{y_p}_{y_p}$$

General solution to the homogeneous version

Proof: Wish to show: $L(y)=g(x)$

Let $y_1, y_2, \dots, y_n$ be a fundamental set for the equation $L(y)=0$

$\Rightarrow L(y_i)=0$

We also know that $L(y_p)=g(x)$

$$L(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p) = c_1 \cancelto{0}{L(y_1)} + c_2 \cancelto{0}{L(y_2)} + \cdots + c_n \cancelto{0}{L(y_n)} + \cancelto{g(x)}{L(y_p)}$$

$$= g(x)$$

Note: To solve a non-homogeneous ODE:

1. Solve the associated homogeneous for a general solution, y_h
2. Find one particular solution, y_p
3. The final solution is the sum of 1 and 2: $y_h + y_p$

EX: Let $y''' - 6y'' + 11y' - 6y = 3x$

a. Show that $y_p = -\frac{11}{12} - \frac{1}{2}x$ is a particular solution

$y_p' = -\frac{1}{2}$

$y_p'' = 0$

$y_p''' = 0$

PLUG IN

$0 - 6(0) + 11\left(-\frac{1}{2}\right) - 6\left(-\frac{11}{12} - \frac{1}{2}x\right)$

$= -\cancel{\frac{11}{2}} + \cancel{\frac{11}{2}} + 3x = 3x$

b. What is the general solution to this equation?

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x$